MATH 6021 Lecture 10 11/16/2020

Recall: Last time, given some "non-trivial" sweepouts $\{\Sigma_t\}$, we can "do" min-max theory to obtain a min-max seq of surfaces

$$\Sigma^n_{t_n} \xrightarrow{\text{as varifold}} V = n_1|\Sigma_1| + \cdots + n_g|\Sigma_g|$$

stationary varifold

where $n_i \in \mathbb{N}$, $\Sigma_i \subset M$ smooth embedded min. hypersurfaces (with singular set of codim > 7 when $\dim M \geq 8$)

Recall: (Yau's Conjecture)

Every closed (M^{n+1}, g), say $n+1 = 3$, contains ∞'ly many smooth closed (embedded) min. hypersurfaces.

Thm: (Almgren-Pitts, Schoen-Simon) $\exists$ at least one min. hypersurface.

Note: Lawson '70 constructed ∞'ly many min. surfaces in (S^3, g_{round}).

In Marques-Neves (Invent. Math. 2017), they proved:

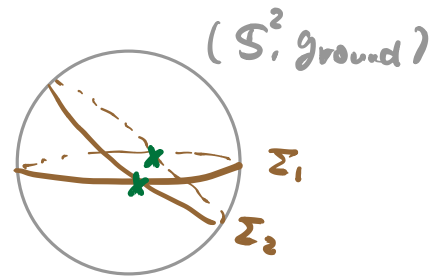
Thm A: Yau's conjecture holds if (M^{n+1}, g) has Ric > 0. (or Frankel property)

Remarks: • when $n+1 \geq 8$, then there may be "small" singular sets.

• when $n+1 = 3$, then $(M^3, g) \overset{\text{diffeo.}}{\sim} S^3/\Gamma$ [Hamilton '82 Ricci Flow.]

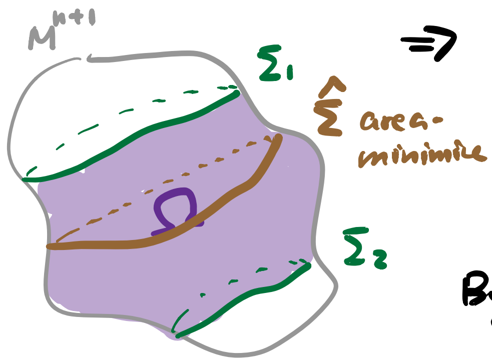
Q: Why Ric > 0 is needed?

A: Ric > 0 $\Rightarrow$ "Frankel Property".



Thm: (Frankel '66) Let (M^{n+1}, g) be closed with $Ric > 0$. Then, any two closed embedded min. hypersurf. Σ_1, Σ_2 in M must intersect somewhere, i.e. $\Sigma_1 \cap \Sigma_2 \neq \phi$.

"Proof": Suppose NOT. Then Σ_1, Σ_2 bounds a region Ω in M.



$\Rightarrow$ Σ_1, Σ_2 are homologous in Ω

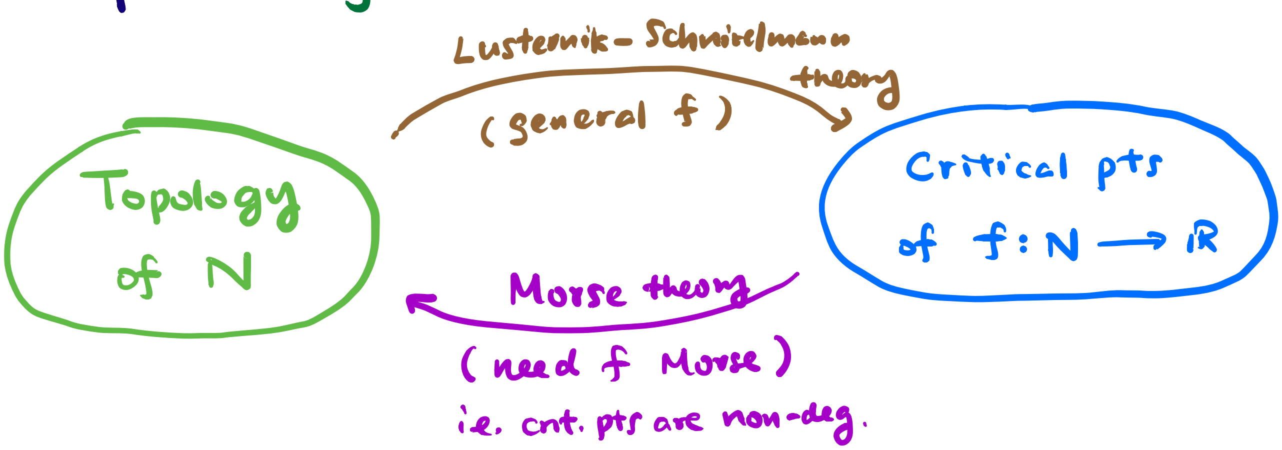
Let $\hat{\Sigma} \subseteq \Omega$ be an area-minimizing hypersurf. in $[\Sigma_1] = [\Sigma_2] \in H_n(\Omega; \mathbb{Z})$

By G.M.T., $\hat{\Sigma}$ is smooth away from small singular set. So $\hat{\Sigma}$ is stable $\leftrightarrow$ contradicts $Ric^M > 0$.

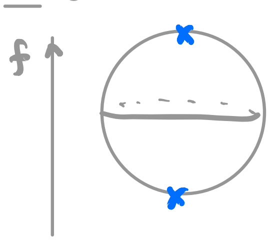
$\square$

Key Approach to Prove Thm A:

Apply Lusternik-Schnirelmann theory & also a "counting argument" to the space of cycles mod 2 (ie with $\mathbb{Z}_2$ coefficients)

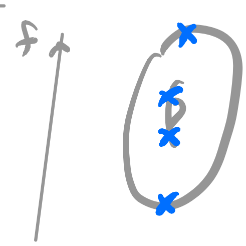


Eg.) $f: S^2 \to \mathbb{R}$



has 2 crit. pts.

Eg.) $f: T^2 \to \mathbb{R}$



has 4 crit. points

Fact: $\exists \bar{f}: T^2 \to \mathbb{R}$ with only 3 crit. points

Our setting: $N \sim$ space of cycles ; $f \sim$ Area functional

Topology of the space of cycles mod 2

Notation: (M^{n+1}, g) closed, orientable Riem. mfd.

$$\mathcal{Z}_k(M;\mathbb{Z}_2) := \{\text{k-dim cycles in } M \text{ mod } 2\}$$ w/. flat/weak topology

Almgren Isomorphism: $\pi_\ell(\mathcal{Z}_k(M;\mathbb{Z}_2)) \cong H_{k+\ell}(M;\mathbb{Z}_2)$

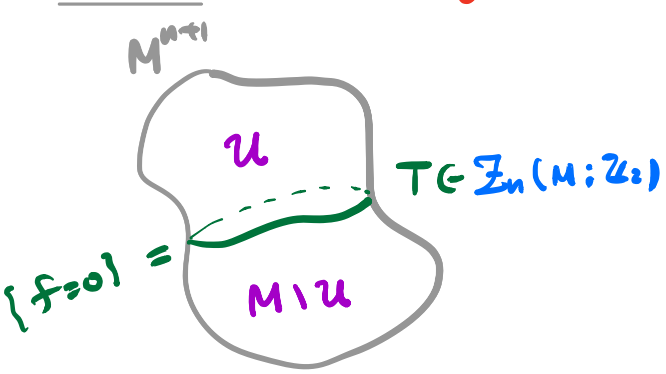
When $k = n = \dim M - 1$, (codim 1 case)

- $\pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2)) \cong H_{n+1}(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$
- $\pi_\ell(\mathcal{Z}_n(M;\mathbb{Z}_2)) \cong H_{n+\ell}(M;\mathbb{Z}_2) = 0$ whenever $\ell \geq 2$

$$\Longrightarrow \quad \mathcal{Z}_n(M;\mathbb{Z}_2) \underset{\substack{\text{homotopic}\\ \text{equivalent}}}{\overset{\text{weakly}}{\sim}} \mathbb{RP}^\infty \ (= B\mathbb{Z}_2)$$

$\Longrightarrow$ $\exists$ homotopically non-trivial k-parameter families of hypersurfaces to do "min-max", like $\mathbb{RP}^k \subseteq \mathbb{RP}^\infty$.

Picture: (Q: why $\mathbb{Z}_2$ coefficients?)



$\partial U = T = \partial(M \setminus U)$ in $\mathbb{Z}_2$-coeff.

$$U \longmapsto M\setminus U$$

$$\begin{tikzcd} I_{n+1}(M;\mathbb{Z}_2) \arrow[rr, "\alpha"] \arrow[dr, "\partial"'] & & I_{n+1}(M;\mathbb{Z}_2) \arrow[dl, "\partial"] \\ & \mathcal{Z}_n(M;\mathbb{Z}_2) & \end{tikzcd}$$

"anti-podal map"

2:1 cover (like $S^n \to \mathbb{RP}^n$)

<u>Gromov '88</u>: regard $\Sigma^n \subset M^{n+1}$ as $\Sigma = \{f=0\}$ for some function $f: M \to \mathbb{R}$.

$$\text{Area}: \boxed{\{f: M \to \mathbb{R}\}/\text{scaling}} \longrightarrow \mathbb{R}_{\geq 0}$$

$\approx$ ∞-dim'l projective space

$$\text{Area}(f) := \text{Area}(\{f=0\}) = \text{Area}(\{\lambda f = 0\}) \quad \forall \text{ constant } \lambda \neq 0.$$

<u>Q</u>: What is the spectrum of this function, defined via Rayleigh quotient? $\rightsquigarrow$ "Volume Spectrum".

Instead of homopoty type of $\mathcal{Z}_n(M;\mathbb{Z}_2)$ as $\mathbb{RP}^\infty$, we will use the structure of the cohomology ring of $\mathbb{RP}^\infty$

cup product structure

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \underbrace{\mathbb{Z}_2[\bar{\lambda}]}_{\substack{\text{polynomial ring} \\ \text{gen. by } \bar{\lambda} \text{ with } \mathbb{Z}_2\text{-coeff.}}} := \{a_0 + a_1\bar{\lambda} + \cdots + a_k\bar{\lambda}^k : a_i \in \mathbb{Z}_2\}$$

where $\bar{\lambda} \in H^1(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is the generator.

<u>Note</u>: $H^k(\mathbb{RP}^\infty;\mathbb{Z}_2) = \{0, \bar{\lambda}^k\} \cong \mathbb{Z}_2$.

<u>Geometric meaning of $\bar{\lambda}$</u>:

A cts family $\gamma: S^1 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$, hence $\gamma \in H_1(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2)$, is a "non-trivial sweepout" $\iff$

$$\underset{H^1}{\bar{\lambda}} \cdot \underset{H_1}{\gamma} = 1 \in \mathbb{Z}_2$$

More generally, we define:

Def$^{\underline{n}}$: Let X be a finite-dim'l simplicial complex (e.g. $X =$ )

A cts map $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is called a "k - sweepout"

if it "detects" the cohomology class $\bar{\lambda}^k \in H^k(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2)$.

i.e. $\boxed{\Phi^*(\bar{\lambda}^k) \neq 0 \in H^k(X;\mathbb{Z}_2)}$

Def$^{\underline{n}}$: For any closed (M^{n+1}, g), we can define, for each $k \in \mathbb{N}$, a geometric invariant called the k-width of (M,g) by

$$\omega_k(M,g) := \inf_{\substack{\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2) \\ k\text{-sweepout}}} \left(\sup_{x \in X} \mathbf{M}(\Phi(x)) \right)$$

Remarks: • The domain X of Φ can vary.

• $(k+1)$-sweepouts are also k-sweepouts. $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$

$\Phi^*(\bar{\lambda}^k) \cup \Phi^*(\bar{\lambda})$

Since $\Phi^*(\bar{\lambda}^{k+1}) \neq 0 \in H^{k+1}(X;\mathbb{Z}_2)$

$$\Rightarrow \quad \Phi^*(\bar{\lambda}^k) \neq 0 \in H^k(X;\mathbb{Z}_2).$$

So, we have a sequence:

$$\omega_1(M,g) \leq \omega_2(M,g) \leq \omega_3(M,g) \leq \cdots \leq \omega_k(M,g) \leq \cdots$$

called the volume spectrum of (M,g).

highly non-linear version of the "spectrum"

and VERY DIFFICULT to compute!!

Example: $(M^{n+1}, g) = (S^3, \bar{g})$ ← round metric

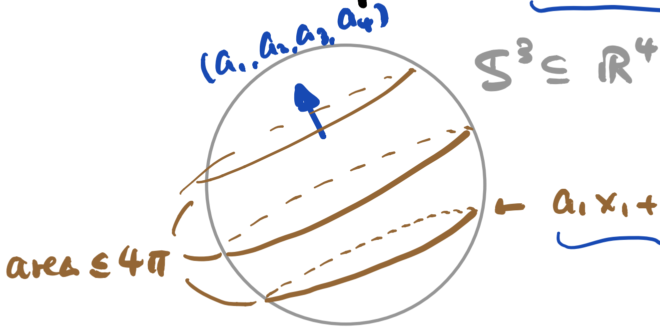
Prop: For $(S^3, \bar{g})$, $\boxed{\omega_1 = \omega_2 = \omega_3 = \omega_4 = 4\pi}$

[Note: 4π = area of totally geodesic $S^2 \subset S^3$.]

"Proof": **Claim:** $\omega_4 \le 4\pi$

$\because \exists$ 4-sweepout $\Phi : \mathbb{RP}^4 \to \mathcal{Z}_2(S^3; \mathbb{Z}_2)$ given by

$$\Phi([\underbrace{a_0 : a_1 : a_2 : a_3 : a_4}]) = \left\{ \begin{array}{l} x = (x_1, x_2, x_3, x_4) \in S^3 \subseteq \mathbb{R}^4 : \\ a_0 + \underbrace{a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4} = 0 \end{array} \right\}$$



← $\underbrace{a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4} = -a_0$

Claim: $4\pi \le \omega_1$ $\qquad$ ($\omega_1 = n\,\text{Area}(\Sigma) \ge 4\pi n$.)

By Min-Max Theory, ω_1 is achieved by some min $\Sigma^2 \subseteq S^3$ (up to integer multiplicities), which has

$$\text{area}(\Sigma) \ge \text{area}(S^2) = 4\pi$$

(by Monotonicity formula)

$\blacksquare$

Thm: $\boxed{\omega_5(S^3, \bar{g}) = 2\pi^2}$ $> 4\pi$

[Note: $2\pi^2$ = area of Clifford torus in S^3.]

Proof is extremely long & difficult, require the solution of the Willmore Conjecture (Marque-Neves '14).

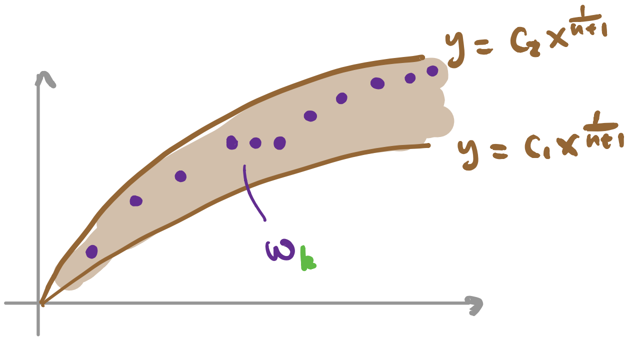
Open Q: compute $\omega_k(S^3, \bar{g})$ for $k \ge 6$?

Although it is very difficult to compute the actual values of ω_k, even for $(S^3, \bar{g})$, one can still study their general asymptotic behavior as $k \to \infty$:

[cf. Weyl asymptotic formula for Δ-spectrum.]

Thm: (Gromov – Guth '09) $\exists\, c_1, c_2 > 0$, depends on (M^{n+1}, g), s.t.

$$c_1 k^{\frac{1}{n+1}} \leq \omega_k(M, g) \leq c_2 k^{\frac{1}{n+1}} \qquad \forall k \in \mathbb{N}$$



i.e. ω_k grows sub-linearly in k

Conjecture (Gromov):

The volume spectrum $\{\omega_k(M^{n+1}, g)\}$ obeys a "Weyl Law" similar in form the classical Weyl law for Δ-spectrum.